

Symmetry of Galvanomagnetic Effects in Antimony*

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The phenomenological theory of galvanomagnetic effects in a medium of point-group symmetry $3m$, isomorphic to antimony, has been developed for the case that these effects can be described in terms of an expansion in powers of the magnetic field. Schemes of coefficients of these powers up to and including the fourth power are constructed explicitly. Various experimental arrangements for a determination of all independent constants are discussed, and the voltages to be expected in each case are worked out in detail. A final section lists the relations between the galvanomagnetic coefficients of the conductivity and resistivity tensors.

1. Introduction

The electrical behavior of antimony is characterized by two important features: Electrical conduction is, for a metal, appreciably anisotropic, and the Hall voltage is not a linear function of the magnetic field H , even for relatively small H . These properties have made it difficult to evaluate the large number of early reports on the galvanomagnetic properties of antimony which just list a 'Hall constant' but do not include a specification of crystal orientation, in addition to giving the Hall voltage as a function of H .

Measurements are in progress at this laboratory to map out the galvanomagnetic effects in single crystals of antimony as a function of orientation of the directions of current and magnetic field with respect to crystal axes. The data have shown that for any given set of these directions the variation of the galvanomagnetic effects as a function of H can be expressed in terms of a rapidly converging power series in H . At room temperature, coefficients of powers up to and including H^3 are of importance, and fourth-order terms possible also play a role.

The symmetry of antimony requires relations between the coefficients of a given power of H for different sets of current and magnetic field directions. In this paper we develop the phenomenological theory of the galvanomagnetic effects in a medium of crystal symmetry isomorphic to the point-group symmetry of antimony, for the special case in which these effects are expressible in a power series in H .

2. The phenomenological description

In anisotropic media the linear relation between current density \mathbf{J} and electric field \mathbf{E} must be described by tensors. With respect to Cartesian coordinate axes, the description has the familiar form

$$J_i = \sigma_{ij} E_j \quad \text{or} \quad E_i = \rho_{ij} J_j. \quad (1)$$

The effect of a magnetic field \mathbf{H} on electrical conduction can be included in (1) by making the conductivity and resistivity tensors general functions of \mathbf{H} . The linear relation between \mathbf{E} and \mathbf{J} is maintained since usually the magnetic forces on electrons in a metal are much larger than the electrical forces (Davis, 1939). σ_{ij} and ρ_{ij} may be general functions of \mathbf{H} , except for the restrictions imposed by the Onsager reciprocity relations which connect otherwise independent irreversible processes. For electrical conduction they take the form (Onsager, 1931):

$$\sigma_{ij}(\mathbf{H}) = \sigma_{ji}(-\mathbf{H}); \quad \rho_{ij}(\mathbf{H}) = \rho_{ji}(-\mathbf{H}). \quad (2)$$

In addition to the restriction expressed by (2), the tensor components will also be subject to the restrictions imposed by the macroscopic or point symmetry of the crystalline material in question. This paper will be concerned with the determination of these relations and of the explicit form of the tensors (2) for crystals with a point-group symmetry $3m$ isomorphic to the point-group symmetry of antimony. In the case in which the dependence of electrical conduction on the magnetic field can be expressed in terms of a rapidly converging expansion according to powers of the magnetic field, the relations existing between the tensor components can be separated into various groups connecting coefficients of the same power of H .

In normal experimental arrangements to determine galvanomagnetic effects one measures voltages produced by a given primary current. To connect directly with experimental results we will carry through the full analysis of the symmetry problem for the resistivity tensor ρ_{ij} . Of course, the same solution applies immediately to the conductivity upon interchange of \mathbf{J} and \mathbf{E} . However, the relations between the coefficients of both tensors are somewhat involved. Since most theoretical discussions are best carried out in terms of the conductivity tensor, we will include a section to exhibit explicitly the relation between these coefficients.

The symmetry condition (2) implies that the resistivity may be written in the form

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$$\varrho_{ij}(\mathbf{H}) = s_{ij}(\mathbf{H}) + a_{ij}(\mathbf{H}), \quad (3)$$

where the two parts have the following symmetry properties:

$$\left. \begin{aligned} s_{ij}(\mathbf{H}) &= s_{ji}(\mathbf{H}) = s_{ij}(-\mathbf{H}), \\ a_{ij}(\mathbf{H}) &= -a_{ji}(\mathbf{H}) = -a_{ij}(-\mathbf{H}). \end{aligned} \right\} \quad (4)$$

If we develop s and a in powers of H , s is an even function, and a an odd function of H :

$$\left. \begin{aligned} s_{ij}(\mathbf{H}) &= \varrho_{ij} + A_{ijkl}H_kH_l + F_{ijklmn}H_kH_lH_mH_n + \dots, \\ a_{ij}(\mathbf{H}) &= R_{ijk}H_k + T_{ijklm}H_kH_lH_m + \dots \end{aligned} \right\} \quad (5)$$

The implied summation in (5) is carried out over all possible values of all repeated indices ($k, l, m, n = 1, 2, 3$).

The symmetry of the current-density and electric-field indices expressed in (4) applies to each coefficient separately. In addition, the convention of summation over the magnetic-field indices adopted in (5) implies that each coefficient is totally symmetric in these indices. The complete symmetry specifications are shown below:

$$\left. \begin{aligned} \varrho_{ij} &= \varrho_{ji}, \\ A_{ijkl} &= A_{jikl} = A_{ij} \text{ (all permutations of } kl), \\ F_{ijklmn} &= F_{jiklmn} = F_{ij} \text{ (all permutations of } klmn), \\ R_{ijk} &= -R_{jik}, \\ T_{ijklm} &= -T_{jiklm} = T_{ij} \text{ (all permutations of } klm). \end{aligned} \right\} \quad (6)$$

Our analysis will be restricted to the coefficients ϱ, A, F, R, T .

3. The number of independent constants

The number of the independent constants for each of the coefficients listed in (6), in the symmetry scheme proper to antimony, is obtained by group character analysis. For this purpose we must determine, first of all, the character of the representation corresponding to the transformation properties of the coefficient in question. This is done most easily by correlating these transformation properties to those of products of vector and tensor components (Fumi, 1952a; Juretschke, 1952). Because of the convention of summation indicated above, all coefficients transform, without extra numerical factors, as follows:

- ϱ_{ij} : symmetric second-order tensor;
- A_{ijkl} : product of two symmetric second-order tensors;
- F_{ijklmn} : product of a symmetric second-order tensor and a totally symmetric fourth-order tensor;
- R_{ijk} : product of two axial vectors;
- T_{ijklm} : product of an axial vector and a totally symmetric third-order tensor.

The characters of the orthogonal transformations belonging to the tensors occurring in this list are

available in general form (Juretschke, 1951; Bhagavan-tam & Venkatarayudu, 1951), and the characters appropriate to the transformation schemes of the coefficients are obtained by constructing the corresponding products.

In Table 1 we list the characters appropriate to the

Table 1. *Characters of the transformation group of the various coefficients of the magnetic field-dependent resistivity in the point group $3m$*

	Classes			No. of independent components
	1	(2)3'	(3)m ⁻	
ϱ_{ij}	6	0	2	2
A_{ijkl}	36	0	4	8
F_{ijklmn}	90	0	6	18
R_{ijk}	9	0	1	2
T_{ijklm}	30	0	2	6

rotation classes of the point group $3m$. Because of the nature of the variables involved, all effects are unchanged under inversion.

The number of independent components for each set of coefficients is obtained directly from the orthogonality relations for characters of irreducible representations. These numbers are listed in the last column of Table 1. They show that the complete description of the effect of a magnetic field on the conduction properties of antimony, up to fourth-power products of the magnetic field components, requires specification of 36 independent constants. To third-order, the description requires 18 constants, while two constants suffice to specify conduction in the absence of a magnetic field.

It may happen, of course, that some of the above constants are numerically equal, or vanish. Non-accidental degeneracy of other coefficients may occur if, for some reason, the internal symmetry applicable to conduction processes is greater than the symmetry of structure.

4. The coefficient schemes

Table 1 lists the number of independent constants for each set of coefficients. The number of non-vanishing constants in a Cartesian coordinate system adapted to the axes of symmetry will, in general, be larger. In this section we derive the complete schemes for the various coefficients with respect to a coordinate system specified as follows: z is parallel to the threefold axis, and the plane of (x, z) is one of the three equivalent planes of reflection.

Transformation from this system of axes, fixed in the crystal, to other axes, particularly axes in the laboratory system, is straightforward, since in the discussion of the summation over magnetic field indices it was specified that all coefficients transform strictly like the corresponding products of coordinates. This rule will, however, introduce numerical factors for some coefficients in the expressions for the coefficient

schemes, particularly for the higher powers of H , since in these expressions repeated equal terms in the summation will appear as a single entry.

The discussion of the construction of the coefficient schemes need be carried out separately for only three cases: ρ and R , and A and T , respectively, have coefficient entries derivable from the same general coefficient scheme.

(A) ρ and R

The tensor form of ρ is well-known:

$$\rho_{ij} = \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & \rho_{11} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix}. \quad (7)$$

The dyadic for R can be constructed directly from (7), since Table 1 indicates that the point symmetry group in question does not distinguish between general and symmetric second-order tensors. The only precaution is that, since R has three indices, with the first two antisymmetric, one must observe the proper correlation of first index of ρ_{ij} and the first index pair of R_{ijk} ($1 \rightarrow 23$, $2 \rightarrow 31$, $3 \rightarrow 12$).

$$R_{(ij)k} = \begin{pmatrix} R_{231} & 0 & 0 \\ 0 & R_{231} & 0 \\ 0 & 0 & R_{123} \end{pmatrix}. \quad (8)$$

Equation (8) is a dyadic between products of electric-field and current-density components on the one hand, and magnetic-field components on the other. The more familiar form of a connection between electric field and current density is obtained by carrying out the summation over magnetic-field components indicated in (5):

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & R_{123}H_3 & -R_{231}H_2 \\ -R_{123}H_3 & 0 & R_{231}H_1 \\ R_{231}H_2 & -R_{231}H_1 & 0 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}. \quad (9)$$

(B) A and T

The discussion of R , above, shows that it is easier to derive the coefficient scheme in the form (8) rather than in the form (9). This is particularly so when higher powers of H are involved. The coefficient schemes for A and T will therefore be derived as a connection between products of electric-field and current-density components and products of magnetic-field components. The direct voltage-current relationship can be constructed without difficulty once the first scheme is known.

In the group $3m$ the general fourth order tensor has 14 independent components. A list of all non-vanishing components has been given by Fumi (1952b). The most direct way of obtaining the coefficient schemes of interest here is to introduce the additional sym-

metry applicable in each case in the general list constructed by Fumi.

A_{ijkl} has symmetry within the first and second index pairs. The introduction of this symmetry reduces the general fourth-order tensor scheme to the form shown in Table 2.

Table 2. *Non-vanishing second-power galvanomagnetic coefficients in the point group 3m*

Each row forms a symmetric component of the resistivity tensor

$E_i, J_j \setminus H_k H_l$	x^2	y^2	z^2	yz	zx	xy
x^2	A_{11}	A_{12}	A_{13}	$-2A_{24}$	0	0
y^2	A_{12}	A_{11}	A_{13}	$2A_{24}$	0	0
z^2	A_{31}	A_{31}	A_{33}	0	0	0
yz	$-A_{42}$	A_{42}	0	$2A_{44}$	0	0
zx	0	0	0	0	$2A_{44}$	$-2A_{42}$
xy	0	0	0	0	$-2A_{24}$	$(A_{11}-A_{12})$

In this table index pairs are labelled by a single index running from 1 to 6.

T_{ijklm} is antisymmetric in the first two indices, and totally symmetric in the last three, and really relates a vector to a third-order tensor. However, by associating one magnetic-field component, say H_k , with the axial vector component $E_i J_j$, T_{ijklm} can be thought of as relating a general tensor of second order to a symmetric tensor of second order. The additional symmetry arising from the total symmetry of the three factors $H_k H_l H_m$ is easily introduced explicitly. In terms of such an array, the coefficient scheme has the form shown in Table 3.

Table 3. *Non-vanishing third-power galvanomagnetic coefficients in the point group 3m*

The sum of the three rows $k = 1, 2, 3$ forms an antisymmetric component of the resistivity tensor.

$(E_i, J_j) H_k$	$H_l H_m$	x^2	y^2	z^2	yz	zx	xy
$(yz)x$	$3T_{112}$	T_{112}	T_{113}	$-2T_{224}$	0	0	0
$(zx)y$	T_{112}	$3T_{112}$	T_{113}	$2T_{224}$	0	0	0
$(xy)z$	T_{331}	T_{331}	T_{333}	0	0	0	0
$(zx)z$	$-T_{224}$	T_{224}	0	$2T_{113}$	0	0	0
$(xy)x$	0	0	0	0	$2T_{331}$	$-2T_{322}$	0
$(yz)y$	0	0	0	0	$-2T_{224}$	$2T_{112}$	0
$(xy)y$	$-T_{322}$	T_{322}	0	$2T_{331}$	0	0	0
$(yz)z$	0	0	0	0	$2T_{113}$	$-2T_{224}$	0
$(zx)x$	0	0	0	0	$-2T_{224}$	$2T_{112}$	0

The three-index notation refers to the first index pair, the third index, and the last index pair, in the manner shown. It should be noted that in Table 3 three rows belong to any one product $E_i J_j$. Furthermore, since the coefficient of $E_i J_j$ is the negative of that of $E_j J_i$, interchanges of components of \mathbf{E} and \mathbf{J} are not listed explicitly.

It is obvious that the scheme of Table 3 can be put into a form similar to (9), since the Hall effect is normal to \mathbf{H} and \mathbf{J} in all powers of H .

Table 4. *Non-vanishing fourth power galvanomagnetic coefficients in the point group 3m*

Each column forms a symmetric component of the resistivity tensor

$H_k H_l H_m H_n \setminus E_i, J_j$	x^2	y^2	z^2	yz	zx	xy
x^4	F_{111}	F_{211}	$3F_{312}$	$F_{412} + 2F_{526}$	0	0
y^4	F_{122}	$F_{111} + F_{211} - F_{122}$	$3F_{312}$	$-(3F_{412} + 2F_{526})$	0	0
z^4	F_{133}	F_{133}	F_{333}	0	0	0
$x^2 y^2$	$F_{111} + 4F_{211} - 3F_{122}$	$F_{111} - 2F_{211} + 3F_{122}$	$6F_{312}$	$6F_{412}$	0	0
$x^2 z^2$	$6F_{223}$	$6F_{123}$	$6F_{313}$	$-6F_{423}$	0	0
$y^2 z^2$	$6F_{123}$	$6F_{223}$	$6F_{313}$	$6F_{423}$	0	0
$x^2 yz$	$12F_{114}$	$6F_{114} - 6F_{124}$	$-12F_{324}$	$12F_{414}$	0	0
$x^2 zx$	0	0	0	0	$12F_{414}$	$3(F_{114} + F_{124})$
$x^2 xy$	0	0	0	0	$8F_{412} + 4F_{526}$	$F_{111} + 2F_{211} - 3F_{122}$
$y^2 yz$	$4F_{124}$	$-6F_{114} - 2F_{124}$	$4F_{324}$	$12F_{414}$	0	0
$y^2 zx$	0	0	0	0	$12F_{414}$	$3(F_{114} + F_{124})$
$y^2 xy$	0	0	0	0	$4F_{526}$	$F_{111} - 2F_{211} + F_{122}$
$z^2 yz$	$-4F_{234}$	$4F_{234}$	0	$4F_{535}$	0	0
$z^2 zx$	0	0	0	0	$4F_{535}$	$-4F_{234}$
$z^2 xy$	0	0	0	0	$-12F_{422}$	$6(F_{223} - F_{123})$

 (C) *F*

The coefficients of F transform like a highly symmetric form of a sixth-order tensor, and this symmetry can be taken into account from the outset in establishing the coefficient scheme. Nevertheless, it is just as easy to introduce this symmetry explicitly in the most general table of components of a sixth-order tensor for the point group $3m$. Such a list has been constructed by Fieschi & Fumi (1953) by systematic application of the results of group theory. If we introduce into their coefficients the pair symmetry of the first two indices and the total symmetry of the last four, we obtain the scheme of coefficients shown in Table 4. The three contracted indices refer to the first, second and third index pair, respectively, and the proper numerical coefficients have been introduced to take into account repeated terms in the summation.

An alternative derivation of such coefficient schemes has been reported by Kao & Katz (1955).

5. The general relations between field and current

Although the coefficient schemes of § 4 contain all the information, it is convenient to have explicit expressions of the field and current relations. These are written down below, up to powers of H^3 .

 (A) *Linear*

The terms linear in H follow directly from (9):

$$\left. \begin{aligned} E_1 &= R_{123}J_2H_3 - R_{231}J_3H_2, \\ E_2 &= -R_{123}J_1H_3 + R_{231}J_3H_1, \\ E_3 &= R_{231}(J_1H_2 - J_2H_1). \end{aligned} \right\} \quad (10)$$

 (B) *Quadratic*

The coefficient scheme of Table 2 yields the quadratic terms:

$$\left. \begin{aligned} E_1 &= J_1(A_{11}H_1^2 + A_{12}H_2^2 + A_{13}H_3^2 - 2A_{24}H_2H_3) \\ &\quad + J_2((A_{11} - A_{12})H_1H_2 - 2A_{24}H_1H_3) \\ &\quad + J_3(2A_{44}H_1H_3 - 2A_{42}H_1H_2), \\ E_2 &= J_1((A_{11} - A_{12})H_1H_2 - 2A_{24}H_1H_3) \\ &\quad + J_2(A_{12}H_1^2 + A_{11}H_2^2 + A_{13}H_3^2 + 2A_{24}H_2H_3) \\ &\quad + J_3(A_{42}(H_2^2 - H_1^2) + 2A_{44}H_2H_3), \\ E_3 &= J_1(2A_{44}H_1H_3 - 2A_{42}H_1H_2) \\ &\quad + J_2(A_{42}(H_2^2 - H_1^2) + 2A_{44}H_2H_3) \\ &\quad + J_3(A_{31}(H_1^2 + H_2^2) + A_{33}H_3^2). \end{aligned} \right\} \quad (11)$$

 (C) *Cubic*

The cubic terms in the voltage-current relation, just like the linear terms, are antisymmetric in E and J . Hence only three combinations of the coefficients in Table 3 occur. These combinations are:

$$\left. \begin{aligned} (yz)(x+y+z) &= 3T_{112}(H_1^2 + H_2^2)H_1 + 3T_{113}H_3^2H_1 \\ &\quad - 6T_{224}H_1H_2H_3, \\ (zx)(x+y+z) &= 3T_{112}(H_1^2 + H_2^2)H_2 + 3T_{113}H_3^2H_2 \\ &\quad + 3T_{224}(H_2^2 - H_1^2)H_3, \\ (xy)(x+y+z) &= 3T_{331}(H_1^2 + H_2^2)H_3 \\ &\quad + T_{322}(H_2^2 - 3H_1^2)H_2 + T_{333}H_3^3. \end{aligned} \right\} \quad (12)$$

From (12) the voltage-current relationship is immediately:

$$\left. \begin{aligned} E_1 &= J_2(3T_{331}(H_1^2 + H_2^2)H_3 + T_{322}(H_2^2 - 3H_1^2)H_2) \\ &\quad + T_{333}H_3^3 \\ &\quad - J_3(3T_{112}(H_1^2 + H_2^2)H_2 + 3T_{113}H_3^2H_2 \\ &\quad + 3T_{224}(H_2^2 - H_1^2)H_3), \\ E_2 &= -J_1(3T_{331}(H_1^2 + H_2^2)H_3 + T_{322}(H_2^2 - 3H_1^2)H_2) \\ &\quad + T_{333}H_3^3 \\ &\quad + J_3(3T_{112}(H_1^2 + H_2^2)H_1 + 3T_{113}H_3^2H_1 \\ &\quad - 6T_{224}H_1H_2H_3), \\ E_3 &= J_1(3T_{112}(H_1^2 + H_2^2)H_2 + 3T_{113}H_3^2H_2 \\ &\quad + 3T_{224}(H_2^2 - H_1^2)H_3) \\ &\quad - J_2(3T_{112}(H_1^2 + H_2^2)H_1 + 3T_{113}H_3^2H_1 \\ &\quad - 6T_{224}H_1H_2H_3). \end{aligned} \right\} \quad (13)$$

(D) Fourth power

The fourth-power relationship can be constructed without difficulty from Table 4. The correlation follows the same rule as the construction of the quadratic terms (11) from Table 2. The general expression will not be written down explicitly.

6. Various experimental arrangements

The conventional galvanomagnetic effects are defined primarily by the direction of \mathbf{H} with respect to \mathbf{J} . Below is a list of the various effects, and the notation which we will use to identify the respective voltages.

(A) H perpendicular to J:

Transverse magneto-resistance: E_J^n .

Hall effect: $-E_{J \times H}$.

Planar Hall effect (Goldberg & Davis, 1954): E_H .

(B) H parallel to J:

Longitudinal magneto-resistance: E_J^n .

In principle all the coefficients defined in the previous sections can be determined by performing the above types of measurements as a function of the directions of \mathbf{J} and \mathbf{H} with respect to the crystallographic axes. In practice, however, this method is not suitable. In the Hall-effect measurements, for instance, it would require, for a given \mathbf{J} , a new set of transverse contacts for each direction of \mathbf{H} with respect to the crystal axes. Furthermore, it has been known for a long time that in anisotropic media the usual distinction between magneto-resistance effects and Hall effects by their dependence on even and odd powers of H , respectively, does not apply directly to the experimental arrangements listed above. The Hall voltage $E_{J \times H}$ usually includes contributions which do not change sign upon reversal of \mathbf{H} ; for comparison with a theory of the Hall effect one must average the quantities $E_{J \times H}(\mathbf{H})$ and $E_{J \times H}(-\mathbf{H})$. Recently it was pointed out that precisely the even-power contributions to the Hall voltage give information about additional galvanomagnetic coefficients in germanium (Mason, Hewitt & Wick, 1953). It is evident, therefore, that it is best to perform a straightforward determination of all galvanomagnetic coefficients without attaching the conventional label to one or another type of measurement.

The usual Hall-effect samples are long, thin cylindrical rods. Thus, the geometry of the sample fixes the direction of \mathbf{J} with respect to the crystal axes. The electrical field \mathbf{E} in the crystal is determined by E_J and any two mutually perpendicular voltages which are normal to \mathbf{J} . For a sample of rectangular cross-section, these voltages are conveniently taken as the voltages normal to the lateral faces.

On a given sample the three voltages are a function of the directions of \mathbf{H} , \mathbf{J} and the lateral faces with respect to the crystal axes. The choice of these directions in an actual experimental arrangement is governed mainly by the desire of obtaining complete and clear-cut information from a relatively small number of samples. In antimony, the two important directions of current flow are obviously normal or parallel to the principal axis. Once the direction of \mathbf{J} is fixed, \mathbf{H} may be either normal or parallel to \mathbf{J} . All of these possibilities will be discussed below. The choice of the direction of the lateral faces is determined primarily by the amount of information one has to obtain from a given sample.

(A) J perpendicular to c, H perpendicular to J

The directions of \mathbf{J} and \mathbf{H} with respect to crystallographic axes are indicated in Fig. 1. θ is the angle

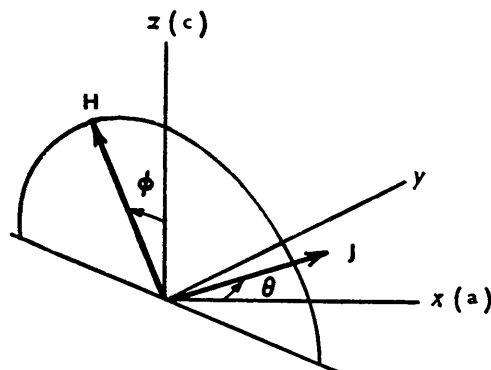


Fig. 1. Specification of the direction of current and magnetic field with respect to crystal axes, for the case \mathbf{J} perpendicular to c , \mathbf{H} perpendicular to \mathbf{J} .

between \mathbf{J} and the x axis. φ is the angle between \mathbf{H} and the z axis in the plane normal to \mathbf{J} , and is positive in the direction $\mathbf{z} \times \mathbf{J}$.

In terms of these angles, the components of \mathbf{J} and \mathbf{H} are:

$$\left. \begin{aligned} J_1 &= J \cos \theta, & H_1 &= -H \sin \theta \sin \varphi, \\ J_2 &= J \sin \theta, & H_2 &= H \cos \theta \sin \varphi, \\ J_3 &= 0, & H_3 &= H \cos \varphi. \end{aligned} \right\} \quad (14)$$

In this arrangement the most convenient directions of the lateral faces are those parallel and normal to the z axis. Therefore, the electric fields of interest are:

$$\left. \begin{aligned} E_J^n &= E_1 \cos \theta + E_2 \sin \theta, \\ E_z &= E_3, \\ E_{z \times J} &= -E_1 \sin \theta + E_2 \cos \theta. \end{aligned} \right\} \quad (15)$$

The fields (15) are determined by substituting the components (14) of \mathbf{H} and \mathbf{J} in the general expressions for the field components (10), (11) and (13) and then forming the combinations indicated:

$$\left. \begin{aligned} E_J^z &= JH^2(A_{12} \sin^2 \varphi + A_{13} \cos^2 \varphi \\ &\quad - 2A_{24} \cos 3\theta \sin \varphi \cos \varphi), \\ E_z &= JHR_{231} \sin \varphi + JH^2 A_{42} \sin 3\theta \sin^2 \varphi \\ &\quad + JH^3(3T_{112} \sin^3 \varphi + 3T_{113} \sin \varphi \cos^2 \varphi \\ &\quad + 3T_{224} \cos 3\theta \sin^2 \varphi \cos \varphi), \\ E_{z \times J} &= -JHR_{123} \cos \varphi + JH^2 2A_{24} \sin 3\theta \sin \theta \cos \varphi \\ &\quad - JH^3(T_{322} \cos 3\theta \sin^3 \varphi \\ &\quad + 3T_{331} \sin^2 \varphi \cos \varphi + T_{333} \cos^3 \varphi). \end{aligned} \right\} (16)$$

For a given sample, θ is fixed, while φ can be varied at will. The various coefficients are separated by an analysis of the H - and φ -dependence of the experimental voltages. By suitable choice of θ , 12 of the 16 galvanomagnetic coefficients can be determined in this manner.

(B) **J parallel to c, H perpendicular to J**

The components of **J** and **H** with respect to crystallographic axes are:

$$\left. \begin{aligned} J_1 &= 0, & H_1 &= H \cos \theta, \\ J_2 &= 0, & H_2 &= H \sin \theta, \\ J_3 &= J, & H_3 &= 0. \end{aligned} \right\} (17)$$

θ is the angle between **H** and the x axis.

In this arrangement there is no advantage in a special orientation of the lateral faces. We will therefore assume an arbitrary orientation. The transverse voltages must then be specified with respect to **H**; a suitable set is given by E_H and $E_{J \times H}$. In terms of the field components with respect to crystallographic axes we have

$$\left. \begin{aligned} E_J^z &= E_3, \\ E_H &= E_1 \cos \theta + E_2 \sin \theta, \\ E_{J \times H} &= -E_1 \sin \theta + E_2 \cos \theta. \end{aligned} \right\} (18)$$

Substitution of the components (17) into (10), (11) and (13) and formation of the combinations indicated in (18) gives:

$$\left. \begin{aligned} E_J &= JH^2 A_{31}, \\ E_H &= -JH^2 A_{42} \sin 3\theta, \\ E_{J \times H} &= JHR_{231} - JH^2 A_{42} \cos 3\theta + 3JH^3 T_{112}. \end{aligned} \right\} (19)$$

The only coefficient not obtained in the first measurement is A_{31} . Transverse voltages are of interest only to confirm the previously obtained values of other coefficients.

(C) **J parallel to H**

In the manner described above all but three of the galvanomagnetic coefficients can be determined. The remaining constants, A_{11} , A_{33} and A_{44} measure longitudinal effects. If θ and φ are the angles indicated in Fig. 2, the components of **J** and **H** are:

$$\left. \begin{aligned} J_1 &= J \cos \theta \sin \varphi, & H_1 &= H \cos \theta \sin \varphi, \\ J_2 &= J \sin \theta \sin \varphi, & H_2 &= H \sin \theta \sin \varphi, \\ J_3 &= J \cos \varphi, & H_3 &= H \cos \varphi. \end{aligned} \right\} (20)$$

For a given sample, φ and θ are fixed, and the only degree of freedom is the reversal of **H**. The most

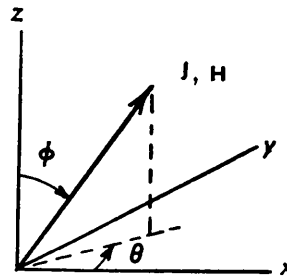


Fig. 2. Specification of the direction of current and magnetic field with respect to crystal axes, for the case **J** parallel to **H**.

convenient directions of the lateral faces, due to the method of preparation, are along $\mathbf{z} \times \mathbf{J}$ and $\mathbf{J} \times (\mathbf{z} \times \mathbf{J})$. The corresponding fields are given by:

$$\left. \begin{aligned} E_J^z &= (E_1 \cos \theta + E_2 \sin \theta) \sin \varphi + E_3 \cos \varphi, \\ E_{z \times J} &= -E_1 \sin \theta + E_2 \cos \theta, \\ E_{J \times (z \times J)} &= -(E_1 \cos \theta + E_2 \sin \theta) \cos \varphi + E_3 \sin \varphi. \end{aligned} \right\} (21)$$

In terms of the components **J** and **H** (21) becomes:

$$\left. \begin{aligned} E_J^z &= JH^2(A_{11} \sin^4 \varphi + (A_{13} + A_{31} + 4A_{44}) \\ &\quad \times \sin^2 \varphi \cos^2 \varphi - 2(A_{42} + A_{24}) \\ &\quad \times \sin 3\theta \sin^3 \varphi \cos \varphi + A_{33} \cos^4 \varphi), \\ E_{z \times J} &= JH(-R_{123} + R_{231}) \sin \varphi \cos \varphi \\ &\quad - JH^2(2A_{24} + A_{42}) \cos 3\theta \sin^2 \varphi \cos \varphi \\ &\quad + JH^3(T_{322} \sin 3\theta \sin^4 \varphi \\ &\quad - 3(T_{331} - T_{112}) \sin^3 \varphi \cos \varphi \\ &\quad - 3T_{224} \sin 3\theta \sin^2 \varphi \cos^2 \varphi \\ &\quad - (T_{333} - 3T_{113}) \sin \varphi \cos^3 \varphi), \\ E_{J \times (z \times J)} &= JH^2(-A_{42} \sin 3\theta \sin^4 \varphi \\ &\quad + (A_{31} + 2A_{44} - A_{11}) \sin^3 \varphi \cos \varphi \\ &\quad + (2A_{24} + A_{42}) \sin 3\theta \sin^2 \varphi \cos^2 \varphi \\ &\quad - (A_{13} + 2A_{24} - A_{33}) \sin \varphi \cos^3 \varphi) \\ &\quad - JH^3(3T_{224} \cos 3\theta \sin^2 \varphi \cos \varphi). \end{aligned} \right\} (22)$$

All necessary information is obtained from E_J^z , and knowledge of the angles θ and φ . $E_{J \times (z \times J)}$ serves as confirmation of the other measurement.

7. The inverse coefficients

The previous sections have discussed the coefficients which are measured in the normal experimental situation: voltages are determined for a given set of **J** and **H**. Theory, however, is primarily concerned with the inverse relationship: current flow is calculated under the condition of a given set of **E** and **J** (for example, see Davis, 1939). In isotropic or highly symmetric media, the connection between the coefficients describing the two kinds of situation is very direct. In the case of antimony the interrelations become more complex.

Equations (3) and (5) define completely the resistivity tensor. We can define in a similar way the conductivity tensor

$$\sigma_{ij}(H) = \sigma_{ij} - P_{ijk}H_k - B_{ijkl}H_kH_l - S_{ijklm}H_kH_lH_m - \dots \quad (23)$$

Except for sign, there is a complete correspondence between the coefficients of resistivity and conductivity, and both must exhibit exactly the same symmetry. In order to establish a connection between the two sets of coefficients we must solve the relation

$$\sigma_{ij} \rho_{jk} = \delta_{ik} \quad (24)$$

for the components of the conductivity tensor, in a power expansion in H up to cubic terms. In these expressions the coefficient of each product of the components of the magnetic field is a combination of the parameters (5) of the resistivity tensor; it corresponds to the coefficient of the same product of the magnetic field components occurring in (23). The results of this straightforward, but lengthy, manipulation are:

$$\left. \begin{aligned} (A) \quad \sigma_{11} &= \frac{1}{\rho_{11}}, & \sigma_{33} &= \frac{1}{\rho_{33}}; \\ (B) \quad P_{123} &= \frac{R_{123}}{\rho_{11}^2}, & P_{231} &= \frac{R_{231}}{\rho_{11}\rho_{33}}; \\ (C) \quad B_{11} &= \frac{A_{11}}{\rho_{11}^2}, & B_{33} &= \frac{A_{33}}{\rho_{33}^2}, \\ & B_{24} = \frac{A_{24}}{\rho_{11}^2}, & B_{42} &= \frac{A_{42}}{\rho_{11}\rho_{33}}, \\ B_{12} &= \frac{A_{12}}{\rho_{11}^2} + \frac{R_{231}^2}{\rho_{11}^2\rho_{33}^2}, & B_{13} &= \frac{A_{13}}{\rho_{11}^2} + \frac{R_{123}^2}{\rho_{11}^3}, \\ B_{31} &= \frac{A_{31}}{\rho_{33}^2} + \frac{R_{231}^2}{\rho_{11}\rho_{33}^2}, \\ B_{44} &= \frac{A_{44}}{\rho_{11}\rho_{33}} - \frac{1}{2} \frac{R_{123}R_{231}}{\rho_{11}^2\rho_{33}}; \\ (D) \quad 3S_{112} &= \frac{3T_{112}}{\rho_{11}\rho_{33}} - \frac{R_{231}}{\rho_{11}\rho_{33}} \left(\frac{A_{12}}{\rho_{11}} + \frac{A_{31}}{\rho_{33}} \right) - \frac{R_{231}^3}{\rho_{11}^2\rho_{33}^2}, \end{aligned} \right\} \quad (25)$$

$$\left. \begin{aligned} 3S_{113} &= \frac{3T_{113}}{\rho_{11}\rho_{33}} - \frac{R_{231}}{\rho_{11}\rho_{33}} \left(\frac{A_{13}}{\rho_{11}} + \frac{A_{33}}{\rho_{33}} \right) \\ &\quad + 2 \frac{R_{123}A_{44}}{\rho_{11}^2\rho_{33}} - \frac{R_{231}R_{123}^2}{\rho_{11}^3\rho_{33}}, \\ 3S_{224} &= \frac{3T_{224}}{\rho_{11}\rho_{33}} - 2 \frac{R_{231}A_{24}}{\rho_{11}^2\rho_{33}} - \frac{R_{123}A_{42}}{\rho_{11}^2\rho_{33}}, \\ 3S_{331} &= \frac{3T_{331}}{\rho_{11}^2} - \frac{R_{123}(A_{11}+A_{12})}{\rho_{11}^3} \\ &\quad + 2 \frac{R_{231}A_{44}}{\rho_{11}^2\rho_{33}} - \frac{R_{123}R_{231}^2}{\rho_{11}^3\rho_{33}}, \\ S_{322} &= \frac{T_{322}}{\rho_{11}^2} + \frac{R_{231}A_{42}}{\rho_{11}^2\rho_{33}}, \\ S_{333} &= \frac{T_{333}}{\rho_{11}^2} - 2 \frac{R_{123}A_{13}}{\rho_{11}^3} - \frac{R_{123}^3}{\rho_{11}^4}. \end{aligned} \right\} \quad (25)$$

These relations indicate that theoretically significant results about the properties of anisotropic media can be obtained, in general, only from a determination of the complete set of experimentally measured coefficients. In particular, it is not obvious that the appearance of, say, cubic terms in the experimental results is necessarily an indication of a cubic effect in the conductivity.

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